

Monogamous nature of symmetric N -qubit states of the W-class: Concurrence and Negativity Tangle

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Using Majorana representation of symmetric N -qubit pure states, we have examined the monogamous nature of the family of states with two-distinct spinors, the W-class of states. We have evaluated the N -concurrence tangle and showed that all the states in this family have vanishing concurrence tangle. The negativity tangle for the W-class of states is shown to be non-zero illustrating the fact that concurrence tangle underestimates the residual entanglement in a pure N -qubit state.

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I. INTRODUCTION

Monogamy of quantum correlations/entanglement, the quantum mechanical feature indicating the restricted shareability of quantum correlations/entanglement among several parties of a composite system, has evoked a lot of interest in the recent years[1 – 28]. The pioneering work of Coffman, Kundu and Wootters [1] has led to a plethora of activity including issues such as monogamy of quantum versus classical correlations [5, 19], monogamy using generalized entropies[12, 15, 16] and monogamy of quantum correlations other than entanglement[18, 21–28]. Quantifying multi-party entanglement is another important issue and measures such as three-tangle (or concurrence-tangle)[1] and negativity tangle[10], based on monogamy inequality, have been proposed for quantifying *residual* or *three-party* entanglement in 3-qubit pure states. In fact, *residual entanglement in a 3-qubit state is defined as the entanglement between the qubits that is not accounted for by the two-qubit entanglement in the state*[1]. The concept of residual entanglement can be generalized to N -party states thus helping in the quantification of N -party entanglement not accounted for by the bipartite entanglement in its subsystems. The nature of monogamy inequality satisfied by N -party states allows one to quantify the N -party residual entanglement in addition to shedding light on the extent of limited shareability of entanglement in the state. In view of the fact [1, 10] that different measures of entanglement give rise to different quantifications of the residual entanglement in 3-qubit pure states, it is natural to expect that similar situation will be realized for N -party states also. While it has been shown that generalized (non-symmetric) W states have vanishing concurrence-tangle[1] indicating only two-way entanglement in them, they are shown to have non-zero negativity tangle[10]. It was thereby concluded that the concurrence-tangle underestimates the residual entanglement in 3-qubit pure states[10].

In this work, we examine the nature of monogamy inequality satisfied by N -qubit pure symmetric states belonging to the W-class. Here the set of all N -qubit symmetric states (invariant under the interchange of qubits) with only two distinct qubits (spinors) characterizing them is defined as the W-class of states, owing to the fact that W states form an integral part of it. We show that the monogamy inequality with square of concurrence[32] as the measure of entanglement holds good with equality for all states of this family (quite similar to the behaviour of W states). With squared negativity-of-partial transpose[33] as measure of entanglement, we examine the monogamous nature of the W-class of states and show that negativity-tangle has non-zero value for all states in this family. We wish to mention here that the Majorana representation of symmetric N -qubit states[29–31] has enabled us to obtain a simplified form for the states with two-distinct spinors thus helping us to obtain the concurrence-tangle and negativity-tangle for the whole family of states.

The article is divided into four parts. Section 1 contains introductory remarks. In Section 2, we make use of the Majorana representation of N qubit pure symmetric states to obtain a simplified form of the states belonging to the W class. We analyze the monogamous nature of the W-class of states in Section 3 and evaluate their concurrence-tangle and negativity-tangle. Section 4 provides a concise summary of the results.

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II. MAJORANA REPRESENTATION OF PURE SYMMETRIC N -QUBIT STATES

In order to examine the nature of monogamy inequality satisfied by N -qubit pure symmetric states of the W-class, we make use of the very elegant Majorana representation [29] of pure symmetric states. While several advantages of using the Majorana representation has been reported in the literature[30, 31], we illustrate here its use in identifying the monogamous nature of symmetric states.

In the Majorana representation [29], a pure symmetric state of N qubits is represented as a *symmetrized* combination of N constituent spinors $|\epsilon_l\rangle$ as

$$|\Psi_{\text{sym}}\rangle = \mathcal{N} \sum_P \hat{P} \{|\epsilon_1, \epsilon_2, \dots, \epsilon_N\rangle\}, \quad (1)$$

where

$$|\epsilon_l\rangle = \cos(\beta_l/2) e^{-i\alpha_l/2} |0\rangle + \sin(\beta_l/2) e^{i\alpha_l/2} |1\rangle, \quad l = 0, 1, 2, \dots, N. \quad (2)$$

Here \hat{P} corresponds to the set of all $N!$ permutations of the spinors (qubits) and \mathcal{N} corresponds to an overall normalization factor.

An N qubit pure symmetric state containing $r(< N)$ distinct spinors $|\epsilon_i\rangle$ ($i = 1, 2, \dots, r$), each repeating n_i times, belongs to the class $\mathcal{D}_{n_1, n_2, \dots, n_r}$ and each degeneracy configuration $\{n_1, n_2, \dots, n_r\}$ (with the numbers n_i being arranged in the descending order) corresponds to a *distinct* SLOCC class[30, 31]. The number of SLOCC inequivalent classes possible for states with r distinct spinors is given by the partition function $p(N, r)$ that gives the distinct possible ways in which the number N can be partitioned into r numbers n_i ($i = 1, 2, \dots, r$) such that $\sum_{i=1}^r n_i = N$ [30, 31]. For instance, a 3-qubit state with only one distinct spinor belongs to the class \mathcal{D}_3 , with two distinct spinors belongs to the class $\{\mathcal{D}_{2,1}\}$ and $\{\mathcal{D}_{1,1,1}\}$ is the class of 3-qubit states with three distinct spinors. The classes \mathcal{D}_3 , $\mathcal{D}_{2,1}$ and $\mathcal{D}_{1,1,1}$ are SLOCC inequivalent and a state belonging to one of these classes cannot be converted into the other (different from itself) by any local operations and classical communications[30, 31]. While the class \mathcal{D}_3 contains only separable states, $\{\mathcal{D}_{2,1}\}$ is the W-class of states and $\{\mathcal{D}_{1,1,1}\}$ corresponds to the GHZ-class of states thus supporting the fact that three qubit pure states can be entangled in two inequivalent ways[34].

A pure symmetric state with 2 distinct spinors belonging to the SLOCC family $\{\mathcal{D}_{N-k,k}, k = 1, 2, \dots, [N/2]\}$ is given by

$$\begin{aligned} |\Psi_{N-k,k}\rangle &= \mathcal{N} \sum_P \hat{P} \{|\underbrace{\epsilon_1, \epsilon_1, \dots, \epsilon_1}_{N-k}; \underbrace{\epsilon_2, \epsilon_2, \dots, \epsilon_2}_k\rangle\} \\ &= \mathcal{N} R_1^{\otimes N} \sum_P \hat{P} \{|\underbrace{0, 0, \dots, 0}_{N-k}; \underbrace{\epsilon'_2, \epsilon'_2, \dots, \epsilon'_2}_k\rangle\}, \end{aligned} \quad (3)$$

where $\epsilon_1 = R_1|0\rangle$ and $\epsilon_2 = R_2|0\rangle$, and

$$|\epsilon'_2\rangle = R_1^{-1} R_2|0\rangle = d_0|0\rangle + d_1|1\rangle, \quad |d_0|^2 + |d_1|^2 = 1, \quad d_1 \neq 0. \quad (4)$$

Thus the symmetric state with two distinct spinors $|\Psi_{N-k,k}\rangle$ is shown to be equivalent, up to local unitary transformations, to

$$|\Psi_{N-k,k}\rangle \equiv \sum_{r=0}^k \sqrt{{N \choose r}} \alpha_r \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle; \quad \alpha_r = \mathcal{N} \frac{(N-r)!}{(N-k)!(k-r)!} d_0^{k-r} d_1^r. \quad (5)$$

It can be seen that $\alpha_r = \delta_{k,r}$ when $d_1 = 1$, $d_0 = 0$ and the state $|\Psi_{N-k,k}\rangle$ reduces to the Dicke state $|\frac{N}{2}, \frac{N}{2} - k\rangle$. It is thus not difficult to see that the states in the family $\mathcal{D}_{N-1,1}$ (with $k = 1$) are SLOCC equivalent to the N -qubit W state $|\frac{N}{2}, \frac{N}{2} - 1\rangle$.

An arbitrary N -qubit pure symmetric state belonging to the W-class is given by

$$|\Psi_{N-1,1}\rangle = \sum_{r=0}^1 \sqrt{{N \choose r}} \alpha_r \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle = \alpha_0 \left| \frac{N}{2}, \frac{N}{2} \right\rangle + \sqrt{N} \alpha_1 \left| \frac{N}{2}, \frac{1}{2} \right\rangle. \quad (6)$$

which may be expressed in terms of standard qubit basis as,

$$|\Psi_{N-1,1}\rangle \equiv a|000 \dots 0\rangle + b \left(\frac{|100 \dots 0\rangle + |010 \dots 0\rangle + \dots + |00 \dots 01\rangle}{\sqrt{N}} \right) \quad (7)$$

with $a = \alpha_0$, $b = \sqrt{N}\alpha_1$ are complex numbers obeying $|a|^2 + |b|^2 = 1$. On taking $a = \cos \frac{\theta}{2}$, $b = \sin \frac{\theta}{2} e^{i\phi}$, ($0 < \theta < \pi$, $0 < \phi < 2\pi$), without any loss of generality and subjecting the N -qubit state (7) to another local unitary transformation $|0\rangle' = |0\rangle$, $|1\rangle' = e^{-i\phi}|1\rangle$ on all the N qubits we obtain a further simplified form

$$|\Psi_{N-1,1}\rangle \equiv \cos \frac{\theta}{2} |000 \cdots 0\rangle + \sin \frac{\theta}{2} \left(\frac{|100 \cdots 0\rangle + |010 \cdots 0\rangle + \cdots + |00 \cdots 01\rangle}{\sqrt{N}} \right) \quad (8)$$

with a single parameter θ , $0 < \theta \leq \pi$ describing the state.

III. MONOGAMOUS NATURE OF PURE SYMMETRIC STATES OF THE W-CLASS: CONCURRENCE- AND NEGATIVITY- TANGLE

Having obtained the simplified form of the N -qubit pure symmetric states with two distinct spinors, we will use it to evaluate the concurrence- and negativity tangle of this family and thereby make a statement about their monogamous nature with respect to different entanglement measures. We carry out this task in the following.

A. Concurrence-tangle:

We start by recalling the monogamy inequality in terms of squared-concurrence in three-qubit systems introduced by Coffman, Kundu and Wootters (CKW) [1]. They[1] have shown that for any 3-qubit pure state Ψ_{ABC} ,

$$C_{AB}^2 + C_{AC}^2 \leq C_{A:BC}^2 \quad (9)$$

where $C_{AB}(C_{AC})$ is the concurrence between A, B (C), while $C_{A:BC} = 2\sqrt{\det \rho_A}$ is the concurrence between system A and BC . The quantity $C_{A:BC}^2 - (C_{AB}^2 + C_{AC}^2)$ is known as *three-tangle* or *concurrence-tangle* and is a measure of three-party entanglement[1]. It was also conjectured[1] that a monogamy relation of the form

$$C_{A_1 A_2}^2 + C_{A_1 A_3}^2 + C_{A_1 A_4}^2 + \cdots + C_{A_1 A_N}^2 \leq C_{A_1:A_2 A_3 A_4 \cdots A_N}^2 \quad (10)$$

holds good for all N -qubit pure states. We can term the quantity

$$C_{A_1:A_2 A_3 A_4 \cdots A_N}^2 - (C_{A_1 A_2}^2 + C_{A_1 A_3}^2 + C_{A_1 A_4}^2 + \cdots + C_{A_1 A_N}^2) \quad (11)$$

as *N-concurrence-tangle*. In fact, it was shown in Ref. [1] that generalized (non-symmetric) 3-qubit W states given by $a|100\rangle + b|010\rangle + c|001\rangle$ have vanishing concurrence-tangle and indicated that their N -qubit counterparts will also exhibit the same feature. We wish to illustrate here that all pure symmetric N -qubit states with two-distinct spinors, the W-class of states, have vanishing N-concurrence-tangle. Towards this end we first wish to evaluate the form of the two-qubit and single-qubit reduced density matrices of the N -qubit state $|\Psi_{N-1,1}\rangle$. Knowing the structure of single qubit density matrices is essential to obtain $C_{A_1:A_2 A_3 A_4 \cdots A_N} = 2\sqrt{\det \rho_A}$ [35], the structure of two-qubit (mixed) density matrices is needed for the evaluation of $C_{A_1 A_2}$. We need to note here that $|\Psi_{N-1,1}\rangle$ being a symmetric state, all its two-qubit and single-qubit subsystems are identical, irrespective of which two qubits or single qubit we choose to consider. That is,

$$\begin{aligned} \rho_{A_1 A_2} &= \rho_{A_1 A_3} = \rho_{A_2 A_3} = \cdots = \rho_{A_{N-1} A_N} \\ \rho_{A_1} &= \rho_{A_2} = \rho_{A_3} = \cdots = \rho_{A_N} \end{aligned} \quad (12)$$

The form of the single-qubit, two-qubit marginals of the state $|\Psi_{N-1,1}\rangle$ for $N = 3, 4, 5, 6$ allows us to generalize and obtain these marginals for any N . In Table 1, we have tabulated the structure of reduced density matrices $\rho_{A_1 A_2}$, ρ_{A_1} of $|\Psi_{N-1,1}\rangle$.

Using the form of two-qubit and single-qubit density matrices given in Table 1, we can readily obtain the structure of the two-qubit and single-qubit density matrices of the N -qubit state $|\Psi_{N-1,1}\rangle$ for any N . We have

$$\rho_{A_1 A_2} = \frac{1}{2N} \begin{pmatrix} 2(N-1+\cos\theta) & \sqrt{N}\sin\theta & \sqrt{N}\sin\theta & 0 \\ \sqrt{N}\sin\theta & 1-\cos\theta & 1-\cos\theta & 0 \\ \sqrt{N}\sin\theta & 1-\cos\theta & 1-\cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (13)$$

TABLE I: The single qubit and two-qubit marginals of $|\Psi_{N-1,1}\rangle$ for $N = 3$ to 6

N	$\rho_{A_1 A_2}$	ρ_{A_1}
3	$\frac{1}{6} \begin{pmatrix} 2(2 + \cos \theta) & \sqrt{3} \sin \theta & \sqrt{3} \sin \theta & 0 \\ \sqrt{3} \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ \sqrt{3} \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{6} \begin{pmatrix} 5 + \cos \theta & \sqrt{3} \sin \theta \\ \sqrt{3} \sin \theta & 1 - \cos \theta \end{pmatrix}$
4	$\frac{1}{8} \begin{pmatrix} 2(3 + \cos \theta) & 2 \sin \theta & 2 \sin \theta & 0 \\ 2 \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ 2 \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{8} \begin{pmatrix} 7 + \cos \theta & 2 \sin \theta \\ 2 \sin \theta & 1 - \cos \theta \end{pmatrix}$
5	$\frac{1}{10} \begin{pmatrix} 2(4 + \cos \theta) & \sqrt{5} \sin \theta & \sqrt{5} \sin \theta & 0 \\ \sqrt{5} \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ \sqrt{5} \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{10} \begin{pmatrix} 9 + \cos \theta & \sqrt{5} \sin \theta \\ \sqrt{5} \sin \theta & 1 - \cos \theta \end{pmatrix}$
6	$\frac{1}{12} \begin{pmatrix} 2(5 + \cos \theta) & \sqrt{6} \sin \theta & \sqrt{6} \sin \theta & 0 \\ \sqrt{6} \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ \sqrt{6} \sin \theta & 1 - \cos \theta & 1 - \cos \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{12} \begin{pmatrix} 11 + \cos \theta & \sqrt{6} \sin \theta \\ \sqrt{6} \sin \theta & 1 - \cos \theta \end{pmatrix}$

and

$$\rho_{A_1} = \frac{1}{2N} \begin{pmatrix} 2N - 1 + \cos \theta & \sqrt{N} \sin \theta \\ \sqrt{N} \sin \theta & 1 - \cos \theta \end{pmatrix} \quad (14)$$

The concurrence[32] of the two-qubit state $\rho_{A_1 A_2}$ is given by $C_{A_1 A_2} = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})$ where λ_i , $i = 1, 2, 3, 4$ are the eigenvalues of the matrix $\rho_{A_1 A_2} \rho'_{A_1 A_2}$, $\rho'_{A_1 A_2} = (\sigma_y \otimes \sigma_y) \rho_{A_1 A_2}^* (\sigma_y \otimes \sigma_y)$ being the spin-flipped density matrix. It can be seen that there is only one non-zero eigenvalue $\lambda = \frac{1}{N^2}(1 - \cos \theta)^2$ for $\rho_{A_1 A_2} \rho'_{A_1 A_2}$ and we therefore have [36]

$$C_{A_1 A_2} = C_{A_1 A_3} = \dots = C_{A_1 A_N} = \sqrt{\lambda} = \frac{1}{N}(1 - \cos \theta). \quad (15)$$

Similarly, we obtain $\det \rho_A = \frac{N-1}{4N^2}(1 - \cos \theta)^2$ and hence

$$C_{A_1:A_2 A_3 \dots A_N}^2 = 4 \det(\rho_A) = \frac{N-1}{N^2}(1 - \cos \theta)^2. \quad (16)$$

As there are $N - 1$ identical two-qubit subsystem density matrices $\rho_{A_1 A_i}$, $i = 2, 3, \dots, N$, with the first qubit being common to all of them, we have

$$C_{A_1 A_2}^2 + C_{A_1 A_3}^2 + C_{A_1 A_4}^2 + \dots + C_{A_1 A_N}^2 = (N - 1)C_{A_1 A_2}^2 = \frac{N-1}{N^2}(1 - \cos \theta)^2 \quad (17)$$

Now, we can readily see that (See Eq.(16))

$$C_{A_1 A_2}^2 + C_{A_1 A_3}^2 + C_{A_1 A_4}^2 + \dots + C_{A_1 A_N}^2 = \frac{N-1}{N^2}(1 - \cos \theta)^2 = C_{A_1:A_2 A_3 A_4 \dots A_N}^2 \quad (18)$$

establishing the relation

$$C_{A_1 A_2}^2 + C_{A_1 A_3}^2 + C_{A_1 A_4}^2 + \cdots + C_{A_1 A_N}^2 = C_{A_1:A_2 A_3 \dots A_N}^2 \quad (19)$$

for the N -qubit pure states of the W-class. Thus in addition to verifying the monogamy inequality, we have shown that *equality* holds good for all N -qubit states belonging to the W-class. In other words, we have shown that the N-concurrence tangle $C_{A_1:A_2 A_3 \dots A_N}^2 - (N-1)C_{A_1 A_2}^2$ vanishes for the family of states $|\Psi_{N-1,1}\rangle$.

B. Negativity tangle:

We begin here by recalling that a monogamy inequality for 3-qubit pure states in terms of negativity-of-partial transpose has been proposed in Ref. [10]. While the vanishing concurrence-tangle for W-states indicated only two-way entanglement, the analogous quantity defined by [10] $\Pi_A = N_{A:BC}^2 - N_{AB}^2 - N_{AC}^2$ showed a non-zero three-way entanglement in W states. While the concurrence-tangle is independent of the focus qubit, the negativity tangle depends on which qubit is considered as the focus qubit. Thus the negativity tangle for 3-qubit pure states is defined as $\Pi = \frac{1}{3}(\Pi_A + \Pi_B + \Pi_C)$ with Π_A, Π_B, Π_C being the negativity tangles with the focus qubits being A, B, C respectively.

Quite similarly to the case of 3-qubit pure states, the negativity tangle for N -qubit pure states is defined as [10]

$$\begin{aligned} \Pi &= \frac{1}{N}(\Pi_1 + \Pi_2 + \dots + \Pi_N) \quad \text{where} \\ \Pi_1 &= N_{A_1:A_2 A_3 \dots A_N}^2 - (N_{A_1 A_2}^2 + N_{A_1 A_3}^2 + \dots + N_{A_1 A_N}^2) \\ \Pi_2 &= N_{A_2:A_1 A_3 \dots A_N}^2 - (N_{A_2 A_1}^2 + N_{A_2 A_3}^2 + \dots + N_{A_2 A_N}^2) \\ &\vdots \\ \Pi_N &= N_{A_N:A_1 A_2 \dots A_{N-1}}^2 - (N_{A_N A_1}^2 + N_{A_N A_2}^2 + \dots + N_{A_N A_{N-1}}^2) \end{aligned} \quad (20)$$

While Ref. [1] indicated vanishing concurrence tangle for N -qubit generalized (non-symmetric) W states, Ref. [10] illustrated that they have a residual N -party entanglement quantified by Π , the negativity tangle. Here, we show that the whole family of pure N -qubit symmetric states belonging to two-distinct spinors (the W-class of states) have non-zero residual entanglement when quantified through negativity tangle. We illustrate this aspect in the following.

Having obtained the two-qubit reduced density matrices of the symmetric state $|\Psi_{N-1,1}\rangle$ (See Eq. (13)), we can readily evaluate their negativity of partial transpose [33]. The partially transposed density matrix of the two-qubit reduced density matrix $\rho_{A_1 A_2}$ obtained in Eq. (13) is evaluated to be

$$\rho_{A_1 A_2}^T = \frac{1}{2N} \begin{pmatrix} 2(N-1+\cos\theta) & \sqrt{N}\sin\theta & \sqrt{N}\sin\theta & 1-\cos\theta \\ \sqrt{N}\sin\theta & 1-\cos\theta & 0 & 0 \\ \sqrt{N}\sin\theta & 0 & 1-\cos\theta & 0 \\ 1-\cos\theta & 0 & 0 & 0 \end{pmatrix} \quad (21)$$

The negativity of partial transpose is given by $(\|\rho_{A_1 A_2}^T\| - 1)/2$ where $\|\rho_{A_1 A_2}^T\|$ is the tracenorm of the partially transposed density matrix $\rho_{A_1 A_2}^T$ and it is the sum of the square-root of eigenvalues of the positive-definite matrix $(\rho_{A_1 A_2}^T)^\dagger \rho_{A_1 A_2}^T$. As the negativity for a two-qubit system varies from 0 to 0.5, we choose to take $N_{A_1 A_2}$ to be

$$N_{A_1 A_2} = \|\rho_{A_1 A_2}^T\| - 1 \quad (22)$$

so that it varies from 0 to 1, quite similar to the variation of concurrence. In fact, this is the convention adopted for negativity in Ref. [10] while obtaining the negativity tangle for three-qubit pure states.

As the negativity of a permutation invariant state is identical for any pair of qubits, we denote $N_{A_1 A_2} = N_{A_1 A_k}$, $k = 2, 3, \dots, N$. Fig.1 shows the plot of negativity $N_{A_1 A_k}$ with respect to θ for the W-class of states $|\Psi_{N-1,1}\rangle$.

It can be seen that with the increase in the number of qubits, the pairwise entanglement quantified by negativity of partial transpose $N_{A_1 A_k}$ decreases quite considerably.

In Ref. [10] it was shown that the negativity between the focus qubit and the remaining two qubits of a pure 3-qubit state matches with their concurrence. The same argument can be extended to N -qubit pure states yielding [37]

$$N_{A_1:A_2 A_3 \dots A_N} = C_{A_1:A_2 A_3 \dots A_N} = 2\sqrt{\det \rho_{A_1}}. \quad (23)$$

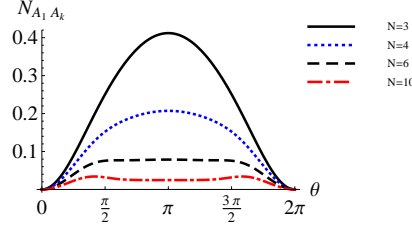


FIG. 1: The plot of $N_{A_1 A_k}$, versus θ in the interval 0 to 2π for arbitrary N qubit state belonging to the W-class.

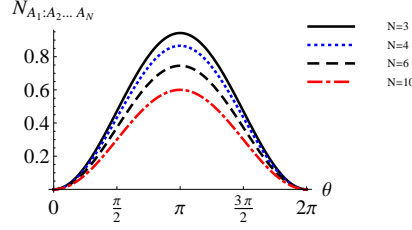


FIG. 2: The plot of $N_{A_1:A_2 A_3...A_N}$, versus θ in the interval 0 to 2π for arbitrary N qubit state belonging to the W-class.

The variation of $N_{A_1:A_2 A_3...A_N}$ with θ , for different values of N , is as shown in Fig. 2. With $\det \rho_{A_1}$ being $\frac{N-1}{4N^2}(1 - \cos \theta)^2$, we have the negativity tangle Π_1 as

$$\begin{aligned} \Pi_1 &= N_{A_1:A_2 A_3...A_N}^2 - (N_{A_1 A_2}^2 + N_{A_1 A_3}^2 + \dots + N_{A_1 A_N}^2) \\ &= \frac{N-1}{N^2}(1 - \cos \theta)^2 - (N-1)N_{A_1 A_2}^2 \end{aligned} \quad (24)$$

But as we are considering symmetric states that are invariant under permutation of qubits, $\Pi_1 = \Pi_2 = \dots = \Pi_N$ and hence,

$$\begin{aligned} \Pi_w &= \frac{\Pi_1 + \Pi_2 + \dots + \Pi_N}{N} = \Pi_1 \\ &= 4 \det \rho_{A_1} - (N-1)N_{A_1 A_k}^2 = (N-1) \left(\frac{(1 - \cos \theta)^2}{N^2} - N_{A_1 A_k}^2 \right) \end{aligned} \quad (25)$$

is the negativity tangle of the state $|\Psi_{N-1,1}\rangle$ belonging to the W-class. We plot a graph of negativity tangle Π_w as a function of θ in Fig 3.

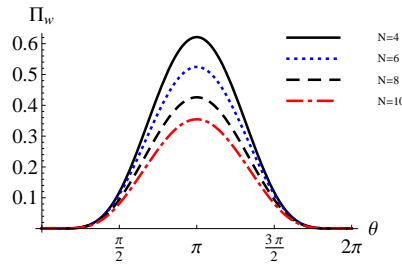


FIG. 3: The plot of negativity tangle Π_w versus θ for the N -qubit symmetric state belonging to the W-class.

In particular, for N -qubit W-states, the negativity-tangle is given by

$$\Pi_w = \frac{N-1}{N^2} \left(4 - \left[\sqrt{(N-2)^2 + 4} - (N-2) \right]^2 \right) \quad (26)$$

Fig. 4 shows the variation of negativity tangle with the number of qubits N for N -qubit W states (corresponding to $\theta = \pi$ in $|\Psi_{N-1,1}\rangle$.)

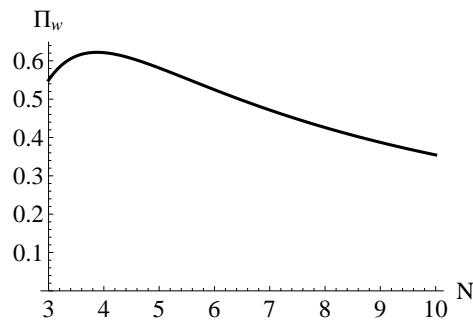


FIG. 4: The negativity-tangle of N -qubit W states as a function of number of qubits N . It can be seen that Π_w is maximum when $N = 4$ and decreases with the number of qubits.

We have thus accomplished the task of evaluating the negativity-tangle for N -qubit pure states belonging to the W-class and illustrated the fact that the concurrence-tangle underestimates the residual entanglement in N -qubit states also. In addition, we have shown that the negativity-tangle which quantifies the residual entanglement in the N -qubit states decreases with increase in N for $N \geq 4$. In fact, as can be seen from Figs. 3 and 4, the three-qubit states belonging to the W-class have lesser residual entanglement than their four-qubit counterparts and for $N \geq 4$, the negativity-tangle goes on decreasing monotonically. Also, one can observe that though the bipartite entanglement $N_{A_1 A_k}$ (See Fig. 1) decreases quite drastically with increase in the number of qubits, the decrease in the residual entanglement Π_w with N is relatively smaller (See Fig. 3). This is due to the slower decrease of the $1 : N - 1$ entanglement, quantified through $N_{A_1 : A_2 A_3 \dots A_N}$, with the increase of qubits (as compared to the fast decrease of $N_{A_1 A_i}$ with N) (See Figs 1 and 2).

IV. CONCLUSION

In this article, we have analyzed the monogamous nature of N -qubit pure states belonging to the W-class using squared concurrence and squared negativity as measures of bipartite entanglement. Using the simplified form of the states belonging to the W-class, obtained using the Majorana representation of N -qubit symmetric pure states, we have evaluated the N -concurrence-tangle and negativity-tangle of this family of states. Quite similar to the N -qubit W-states, we have shown that all states in the W-class of states have vanishing concurrence-tangle. By showing that W-class of states have non-zero negativity-tangle, we have proved the fact that concurrence-tangle underestimates the residual entanglement even in N -qubit states with $N \geq 3$. It would be of interest to examine the nature of monogamy inequality in N -qubit symmetric states belonging to different SLOCC inequivalent families and compare their residual entanglement.

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- [35] Though the concurrence is defined only for two-qubit systems, the N qubit state being pure the $N - 1$ -qubits essentially belong to a two-dimensional space and hence one can define the concurrence between a single qubit and the remaining $N - 1$ qubits[1]. Also, the effective concurrence is the concurrence between two-qubits in a pure state leading to $C_{A_1:A_2A_3A_4\ldots A_N} = 2\sqrt{\det \rho_A}$.
- [36] Notice here that the concurrence between any two qubits becomes maximum and is equal to $2/N$ when $\theta = \pi$ for the W-states. In fact this is the maximum bipartite entanglement in an N -qubit state, achievable for W-states, as is shown in Ref. [2].
- [37] The authenticity of the relation $N_{A_1:A_2A_3\ldots A_N} = C_{A_1:A_2A_3\ldots A_N}$ is explicitly verified for $N = 3, 4, 5, 6$ and generalized thereby to any N .